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On canonical Cohen–Macaulay modules

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ABSTRACT

Let $(R,\mathfrak{m})$ be a Noetherian local ring which is a homomorphic image of a local Gorenstein ring and let M be a finitely generated R-module of dimension $d>0$. According to Schenzel (2004) [Sc3], M is called a *canonical Cohen–Macaulay* module (CCM module for short) if the canonical module $K(M)$ of M is Cohen–Macaulay. We give another characterization of CCM modules. We describe the non-canonical Cohen–Macaulay locus $\mathrm{nCCM}(M)$ of M. If $d \leqslant 4$ then $\mathrm{nCCM}(M)$ is closed in $\mathrm{Spec}(R)$. For each $d \geqslant 5$ there are reduced geometric local rings R of dimension d such that $\mathrm{nCCM}(R)$ is not stable under specialization.

1. Introduction

We first fix a few notations and recall a few notions.

Notation 1.1. Throughout this paper, $(R,\mathfrak{m})$ is a Noetherian local ring which is a quotient of an n-dimensional local Gorenstein ring $(R',\mathfrak{m}')$. Let M be a finitely generated R-module with $\dim M = d$. For each $i \in \mathbb{N}_0$, let $K_R^i(M) = K^i(M)$ denote the i-th *deficiency module* of M that is the finitely generated R-module $\mathrm{Ext}_{R'}^{n-i}(M,R')$. Observe that the formation of deficiency modules is base-ring independent in the following sense: if $\mathfrak{a} \subset R$ is an ideal such that $\mathfrak{a}M = 0$, then the R-modules $K_R^i(M)$ and $K_{R/\mathfrak{a}}^i(M)$ are equal for each $i \in \mathbb{N}_0$. If $E(R/\mathfrak{m})$ denotes the injective envelope of $R/\mathfrak{m}$, the Local

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Duality Theorem gives an isomorphism $H^i_{\mathfrak{m}}(M) \cong \mathrm{Hom}_R(K^i(M), E(R/\mathfrak{m}))$ for all $i \in \mathbb{N}_0$. By $K(M)$ we denote the *canonical module* $K^d(M)$ of M. Throughout our paper, we use the standard convention that $\mathrm{depth}_R(0) = \infty > 0$.

In Section 2 we shall prove a lifting result for the CCM-property of M and deduce two new characterizations of CCM modules. To make this more precise, let us recall, that Schenzel [Sc3] did prove that for $d > 0$, the property of being a CCM module on M is inherited by M/xM if $x \in \mathfrak{m}$ is a so-called strict f-element with respect to M (see Definition 2.2). Our first main result proves that for $d \geqslant 4$ the CCM-property also lifts from M/xM to M for such elements x (see Theorem 2.5). For $d = 3$, this lifting property need not hold any more (see Remark 2.8). As a consequence of this we get that for $d \geqslant 3$ and for each so-called strict f-sequence $x_1, \ldots, x_{d-3}$ with respect to M (see Definition 2.2) the module M is CCM if and only if $K^2(M/\sum_{i=1}^{d-3} x_i M)$ is of depth > 0 – which includes the case that the latter module vanishes (see Corollary 2.6). As a further consequence we get that a generalized Cohen–Macaulay module of dimension $\geqslant 3$ is CCM if and only if the local cohomology modules $H^i_{\mathfrak{m}}(M)$ vanish for all $i = 2, \ldots, d-1$ or – equivalently – the $\mathfrak{m}$-transform $D_{\mathfrak{m}}(M)$ of M is a (finitely generated) Cohen–Macaulay module (see Corollary 2.7).

In Section 3 we study the relation between the CCM-property and the polynomial type $p(M)$ of the module M (see Reminder 3.1). We first consider the case $p(M) = 1$ and prove that in this situation M is CCM if and only if it satisfies the two equivalent conditions, in which $M_{[1]} \subseteq M$ denotes the largest submodule of dimension $\leqslant 1$ (see Proposition 3.7):

(ii) $H^i_{\mathfrak{m}}(M) = 0$ for all $i = 3, \ldots, d-1$ and $K^2(M)$ is 0 or Cohen–Macaulay of dimension 1.
(iii) The ideal transform $D_{\mathfrak{a}}(M^{[1]})$ of $M^{[1]} := M/M_{[1]}$ with respect to $\mathfrak{a} := (0 :_R H^2_{\mathfrak{m}}(M))$ is a (finitely generated) Cohen–Macaulay module.

In a second step, we also admit that $p(M) > 1$. This will lead us to another characterization of CCM modules, which involves once more strict f-sequences (see Theorem 3.10): namely, if $d \geqslant 3$, if the polynomial type $p(M) =: k$ of M is positive and if $x_1, \ldots, x_{k-1}$ is a strict f-sequence with respect to M, then M is CCM if and only if either

(a) $k \leqslant d-2$, $K^i(M) = 0$ for $i = k+2, \ldots, d-1$ and $K^2(M/\sum_{j=1}^{k-1} x_j M)$ is 0 or Cohen–Macaulay of dimension 1; or
(b) $k = d-1$ and $K^2(M/\sum_{j=1}^{d-3} x_j M)$ is of dimension 2 and of depth > 0.

In Section 4 we finally study the non-CCM-locus of M, which is defined as the set of primes $\mathrm{nCCM}(M) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \text{ is not CCM}\}$. We show that under a certain "unmixedness" condition on the support $\mathrm{Supp}_R M$ of M, the set $\mathrm{nCCM}(M)$ coincides with the non-CM-locus $\mathrm{nCM}(K(M))$ and hence with the union of all pseudo supports $\mathrm{Psupp}^i(K(M))$ of the canonical module $K(M)$ of M (see Proposition 4.4). It follows in particular that $\mathrm{nCCM}(M)$ is a closed set if $d \leqslant 4$. We also compare the locus $\mathrm{nCCM}(M)$ with the union of all pseudo supports of the canonical modules of the components of the dimension filtration of M (see Proposition 4.6). Finally, on use of a construction found in Evans–Griffith [EG] we show that for each $d \geqslant 5$ and each field K there is a reduced local ring of $(R, \mathfrak{m})$ of dimension d, essentially of finite type over K, satisfying $R/\mathfrak{m} \cong K$ and such that $\mathrm{nCCM}(R)$ consists of single prime $\mathfrak{p}$ which is in addition different from $\mathfrak{m}$ (see Proposition 4.9). Hence $\mathrm{nCCM}(R)$ is not stable under specialization in this case (see Corollary 4.10).

2. Strict f-sequences and canonical Cohen–Macaulay modules

Definition 2.1. (See [Sc3, Definition 3.1].) The finitely generated R-module M is called *canonical Cohen–Macaulay* (CCM for short) if the canonical module $K(M)$ of M is Cohen–Macaulay (CM for short).

In view of the previously mentioned base-ring independence of deficiency modules, the property of being CCM is also base-ring independent. More precisely, if $\mathfrak{a} \subset R$ is an ideal such that $\mathfrak{a}M = 0$, then M is CCM as an R-module if and only if it is as an $R/\mathfrak{a}$-module.

There is a number of sufficient conditions for a module M to be CCM. If $\dim M \leqslant 2$ then M is CCM. It is clear that any CM module is CCM. The concept of *sequentially Cohen–Macaulay module* (sequentially CM module for short) was introduced by Stanley [St, p. 87], for graded modules. This notion was studied in the local case starting with the work of Schenzel [Sc2]. It is easy to see that any sequentially CM module is CCM. The notion of *pseudo Cohen–Macaulay module* (pseudo CM module for short) was introduced in [CN]. By [CN, Theorem 3.1] the module M is pseudo CM if and only if M/N is CM, where $N \subset M$ is the largest submodule of dimension less than d. As $K(M) \cong K(M/N)$, any pseudo CM module is CCM.

Our first main result shows that for certain elements $x \in R$ (under an obvious restriction on the dimension), the R-module M is CCM if and only if M/xM is. That the CCM-property of M implies the CCM-property of M/xM was actually proved already by Schenzel [Sc3, 3.3(b)].

Definition 2.2. (See [CMN1].) Following I.G. Macdonald [Mac], we denote the set of attached primes of an Artinian R-module A by $\operatorname{Att}_R A$. An element $x \in \mathfrak{m}$ is called a *strict filter regular element* (strict f-element for short) with respect to M if $x \notin \mathfrak{p}$ for all

$$\mathfrak{p} \in \bigcup_{i=1}^{d} \operatorname{Att}_R H^i_{\mathfrak{m}}(M) \setminus \{\mathfrak{m}\}.$$

A sequence $x_1, \ldots, x_t$ of elements in $\mathfrak{m}$ is called a *strict filter regular sequence* (strict f-sequence for short) with respect to M if x_{j+1} is a strict f-element with respect to the R-module $M/(x_1, \ldots, x_j)M$ for all $j = 0, \ldots, t-1$.

Note that for each integer $t > 0$, by Prime Avoidance, there always exists a strict f-sequence of length t with respect to M.

Remark 2.3. (a) By [BS, 11.3.3] we have $\operatorname{Ass}_R M \subseteq \bigcup_{i=0}^{d} \operatorname{Att}_R H^i_{\mathfrak{m}}(M)$. Therefore, if x is a strict f-element with respect to M, it is a filter regular element (f-element for short) with respect to M, i.e. $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R M \setminus \{\mathfrak{m}\}$. Hence each strict f-sequence with respect to M is a filter regular sequence (f-sequence for short) with respect to M in sense of Cuong–Schenzel–Trung [CST].

(b) By [S, Theorem 2.3], we have $\operatorname{Ass}_R K^i(M) = \operatorname{Att}_R H^i_{\mathfrak{m}}(M)$ for all i. Therefore x is a strict f-element with respect to M if and only if x is an f-element with respect to $K^i(M)$ for all $i \geqslant 0$, thus if and only if $\ell(0 :_{K^i(M)} x) < \infty$ for all $i \geqslant 0$. In particular, if $d > 0$ and x is a strict f-element with respect to M then x is $K(M)$-regular.

We begin with the following auxiliary result. The sequence occurring in statement (a) may be found already in [Sc3].

Lemma 2.4. *Let $x \in \mathfrak{m}$ and let $i \in \mathbb{N}_0$.*

(a) *If x is a strict* f*-element with respect to M, there is an exact sequence*

$$0 \to K^{i+1}(M)/xK^{i+1}(M) \to K^i(M/xM) \to (0 :_{K^i(M)} x) \to 0.$$

(b) *If x is a strict* f*-element with respect to M, then $K^i(M/xM) = 0$ if and only if $K^{i+1}(M) = 0$ and x is $K^i(M)$-regular.*
(c) *If x is an* f*-element with respect to M such that* $\operatorname{depth}(M/xM) > 0$, *then x is M-regular.*
(d) *If x is a strict* f*-element with respect to M such that* $\operatorname{depth}(K^i(M/xM)) > 0$, *then* $\operatorname{depth}(K^{i+1}(M)) > 1$.
(e) *If x is a strict* f*-element with respect to M such that either* $\dim K^{i+1}(M) > 0$ *or* $\dim K^i(M/xM) > 0$, *then* $\dim K^i(M/xM) = \dim K^{i+1}(M) - 1$.

Proof. (a): We form the $K^*(\bullet) = \mathrm{Ext}_{R'}^{n-*}(\bullet, R')$-sequence associated to the exact sequence $0 \to M/(0 :_M x) \xrightarrow{\iota} M \to M/xM \to 0$, where ι is induced by $x\mathrm{Id}_M$. This gives an exact sequence $0 \to \mathrm{Coker}(K^{i+1}(\iota)) \to K^i(M/xM) \to \mathrm{Ker}(K^i(\iota)) \to 0$. Consider the short exact sequence $0 \to (0 :_M x) \to M \xrightarrow{\varrho} M/(0 :_M x) \to 0$, in which ϱ is the canonical map. As x is an f-element with respect to M, the module $(0 :_M x)$ is of finite length, so that $K^j(0 :_M x) = 0$ for all $j > 0$. This shows that $K^j(\varrho) : K^j(M/(0 :_M x)) \to K^j(M)$ is an isomorphism for $j > 0$ and a monomorphism for $j = 0$. As $\iota \circ \varrho = x\mathrm{Id}_M$ our claim follows.

(b): This follows immediate from statement (a) on use of Nakayama.

(c): As x is an f-element with respect to M we have $xM \cap H^0_{\mathfrak{m}}(M) = x(H^0_{\mathfrak{m}}(M) :_M x) = xH^0_{\mathfrak{m}}(M)$. So, by Nakayama, $H^0_{\mathfrak{m}}(M) \subseteq xM$ implies that $H^0_{\mathfrak{m}}(M) = 0$. This proves our claim as $\mathrm{depth}(M/xM) > 0$ implies that $H^0_{\mathfrak{m}}(M) \subseteq xM$.

(d): By statement (a) we get $\mathrm{depth}(K^{i+1}(M)/xK^{i+1}(M)) > 0$. As x is an f-element with respect to $K^{i+1}(M)$, our claim follows on use of statement (c).

(e): This is clear by statement (a) as $(0 :_{K^i(M)} x)$ is of finite length. □

Now, we can give the announced first main result.

Theorem 2.5. *Assume that* $\dim M = d \geqslant 4$ *and let* $x \in \mathfrak{m}$ *be a strict* f*-element with respect to* M*. Then* M *is* CCM *if and only if* M/xM *is.*

Proof. By [Sc3, 3.3(b)] we know that $M_1 := M/xM$ is CCM if M is CCM.

Conversely, assume that M_1 is CCM. As x is an f-element with respect to M we have $\dim(M_1) = d-1$. Let $y \in \mathfrak{m}$ be a strict f-element with respect to M_1. Since M_1 is CCM, the module $K^{d-1}(M_1) = K(M_1)$ is CM of dimension $d-1$. As y is a strict f-element with respect to M_1, it is $K(M_1)$-regular and so $K^{d-1}(M_1)/yK_{d-1}(M_1)$ is CM of dimension $d-2 \geqslant 2$. As $\dim(M_1/yM_1) = d-2 \geqslant 2$ we have $\mathrm{depth}(K^{d-2}(M_1/yM_1)) = \mathrm{depth}(K(M_1/yM_1)) \geqslant 2$. So, by the exact sequence of Lemma 2.4(a), applied with M_1, y and $d-2$ instead of M, x and i respectively, the module $(0 :_{K^{d-2}(M_1)} y)$ has depth > 0 and hence vanishes. It follows that y is $K^{d-2}(M_1)$-regular and thus $K^{d-2}(M_1)$ is of positive depth.

Our next claim is that x is $K^{d-1}(M)$-regular. If $K^{d-2}(M/xM) = K^{d-2}(M_1) = 0$, this is immediate by 2.4(b). So, let $K^{d-2}(M_1) \neq 0$. As $K^{d-2}(M_1)$ is of positive depth, the sequence of 2.4(a), applied with $i = d-2$, shows that $K^{d-1}(M)/xK^{d-1}(M)$ is of positive depth, too. On application of 2.4(c) with $K^{d-1}(M)$ instead of M it follows that x is indeed $K^{d-1}(M)$-regular.

Now, if we apply the exact sequence of 2.4(a) with $i = d-1$ we get an isomorphism $K(M_1) \cong K(M)/xK(M)$. Since $K(M_1)$ is CM and x is $K(M)$-regular, $K(M)$ is CM, and hence M is CCM. □

Corollary 2.6. *Suppose that* $d \geqslant 3$ *and let* $x_1, \dots, x_{d-3}$ *is a strict* f*-sequence with respect to* M*. Then* M *is* CCM *if and only if* $\mathrm{depth}(K^2(M/\sum_{i=1}^{d-3} x_i M)) > 0$.

Proof. We proceed by induction on d. First, let $d = 3$. Let $D^\bullet$ be a dualizing complex of R and let $C^\bullet(M)$ be the corresponding complex of deficiency of M, as defined by Schenzel (see [Sc3, Definition 4.1]). Then by [Sc3, Proposition 4.2], the 3-dimensional module M is CCM if and only if the cohomology module $H^2(\mathrm{Hom}(C^\bullet(M), D^\bullet))$ vanishes. As in the proof of [Sc3, Proposition 4.2, p. 760], we get an isomorphism

$$K^0\big(K^2(M)\big) \cong H^0\big(\mathrm{Hom}\big(K^2(M), D^\bullet\big)\big) \cong H^2\big(\mathrm{Hom}\big(C^\bullet(M), D^\bullet\big)\big),$$

so that M is CCM if and only if $\mathrm{depth}\, K^2(M) > 0$.

Now let $d > 3$. By Theorem 2.5, M is CCM if and only if $M_1 := M/x_1M$ is. Observe that $x_2, \dots, x_{d-3}$ is a strict f-sequence with respect to M_1. So, by induction M_1 is CCM if and only if $K^2(M_1/\sum_{i=2}^{d-3} x_i M_1) = K^2(M/\sum_{i=1}^{d-3} x_i M)$ is of depth > 0. Altogether, this completes our proof. □

Recall that according to [CST] the R-module M is said to be *generalized Cohen–Macaulay* (generalized CM for short) if $H^i_{\mathfrak{m}}(M)$ is of finite length for all $i < \dim M$. For each ideal $\mathfrak{a} \subseteq R$ we use $D_{\mathfrak{a}}(M)$ to denote the $\mathfrak{a}$-transform $\varinjlim_n \operatorname{Hom}_R(\mathfrak{a}^n, M)$ of M. As a further application of Theorem 2.5 we get the following criterion for the CCM-property of generalized CM modules.

Corollary 2.7. *Suppose that $d \geqslant 3$ and M is generalized* CM. *Then the following statements are equivalent.*

(i) *M is* CCM.
(ii) *$H^i_{\mathfrak{m}}(M) = 0$ for all $i = 2, \dots, d-1$.*
(iii) *$D_{\mathfrak{m}}(M)$ is a (finitely generated)* CM *module.*

Proof. (i) $\Leftrightarrow$ (ii): As M is generalized CM, the modules $K^i(M)$ are of finite length for $0 \leqslant i < d$. Now Schenzel's result [Sc1, Korollar 1.4] – which holds also if the ring R is replaced by the R-module M – yields isomorphisms

$$H^i_{\mathfrak{m}}\big(K(M)\big) \cong K^0\big(K^{d-i+1}(M)\big), \quad \text{for all } i = 2, \dots, d-1.$$

As $H^i_{\mathfrak{m}}(K(M)) = 0$ for $i = 1, 2$ and by local duality it follows, that $K(M)$ is CM if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all $i = 2, \dots, d-1$. This proves the stated equivalence.

(ii) $\Leftrightarrow$ (iii): This is clear by the short exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to D_{\mathfrak{m}}(M) \to H^1_{\mathfrak{m}}(M) \to 0$$

and the relations $H^i_{\mathfrak{m}}(D_{\mathfrak{m}}(M)) = 0$ for $i = 0, 1$ and $H^j_{\mathfrak{m}}(D_{\mathfrak{m}}(M)) \cong H^j_{\mathfrak{m}}(M)$ for all $j > 1$ (see [BS, Chapter 2]). $\square$

Remark 2.8. Note that the conclusion of Theorem 2.5 is not true for $d = 3$. In fact, let M be a generalized CM module with $\dim M = 3$ and $H^2_{\mathfrak{m}}(M) \neq 0$. (For example take $M = R := A_{A_+}$, where K is a field, $B := K[x, y, z]/(x^3 + y^3 + z^3)$ and A is the Segre product ring $B \times B = \bigoplus_{n \in \mathbb{N}_0} B_n \otimes_K B_n$.) Then M is not CCM by Corollary 2.6. However, since $\dim(M/xM) = 2$ for any strict f-element x with respect to M, it follows that M/xM is CCM.

3. Polynomial type and canonical Cohen–Macaulay modules

Reminder 3.1. The concept of polynomial type was introduced by N.T. Cuong [C]. Let $\underline{x} = (x_1, \dots, x_d)$ be a system of parameters for M and $\underline{n} = (n_1, \dots, n_d)$ a family of positive integers. Set

$$I_{M,\underline{x}}(\underline{n}) := \ell\big(M/\big(x_1^{n_1}, \dots, x_d^{n_d}\big)M\big) - n_1 \dots n_d e(\underline{x}; M),$$

where $e(\underline{x}; M)$ is the multiplicity of M with respect to $\underline{x}$. Consider $I_{M,\underline{x}}(\underline{n})$ as a function in $\underline{n}$. In general this function is not a polynomial for $n_1, \dots, n_d \gg 0$, but it always takes non-negative values for $n_1, \dots, n_d \gg 0$ and is bounded from above by polynomials. Especially, the least degree of all polynomials in $\underline{n}$ which bound from above the function $I_{M,\underline{x}}(\underline{n})$ is independent of the choice of $\underline{x}$ (see [C, Theorem 2.3]). This least degree is called the *polynomial type* of M and denoted by $p(M)$.

Remark 3.2. If we stipulate that the degree of the zero polynomial is $-\infty$, then M is CM if and only if $p(M) = -\infty$. Moreover M is generalized CM if and only if $p(M) \leqslant 0$. In general

$$p(M) = \max_{0 \leqslant i \leqslant d-1} \dim\big(R/\operatorname{Ann}_R H^i_{\mathfrak{m}}(M)\big) = \max_{0 \leqslant i \leqslant d-1} \dim K^i(M) \leqslant d-1$$

(see [C, Theorem 3.1(i)] and observe that by Local Duality $\operatorname{Ann}_R H^i_{\mathfrak{m}}(M) = \operatorname{Ann}_R K^i(M)$). If M is equidimensional then $p(M) = \dim \operatorname{nCM}(M)$, where $\operatorname{nCM}(M)$ is the non-CM-locus of M (see [C, Theorem 3.1(ii)]).

Lemma 3.3. *(See [CMN, Lemma 3.1].) Let $x \in \mathfrak{m}$ be a strict* f*-element with respect to M. If $p(M) > 0$ then $p(M/xM) = p(M) - 1$.*

In this section, we give a characterization of CCM modules which depends on the polynomial type. First we consider the case where $p(M) = 1$. In this case, a certain ideal transform will play a crucial role, and so we first prove the two following auxiliary results.

Lemma 3.4. *Set $\mathfrak{a} := (0 :_R H^2_{\mathfrak{m}}(M))$. Assume that $\ell(H^1_{\mathfrak{m}}(M)) < \infty$ and $\dim(R/\mathfrak{a}) \leqslant 1$. Then $D_{\mathfrak{a}}(M)$ is a finitely generated R-module such that $H^i_{\mathfrak{m}}(D_{\mathfrak{a}}(M)) = 0$ for $i = 0, 1$ and the R-module $H^2_{\mathfrak{m}}(D_{\mathfrak{a}}(M))$ is of finite length. Moreover, $H^i_{\mathfrak{m}}(D_{\mathfrak{a}}(M)) \cong H^i_{\mathfrak{m}}(M)$ for all $i \geqslant 3$.*

Proof. According to our hypothesis we have

$$f^{\mathfrak{a}}_{\mathfrak{m}}(M) := \inf\bigl\{i \in \mathbb{N}_0 \bigm| \mathfrak{a} \nsubseteq \operatorname{rad}\bigl(0 :_R H^i_{\mathfrak{m}}(M)\bigr)\bigr\} \geqslant 3.$$

As $\operatorname{ht}((\mathfrak{p} + \mathfrak{m})/\mathfrak{p}) = \dim(R/\mathfrak{p})$ for each $\mathfrak{p} \in \operatorname{Spec}(R)$, it follows by [BS, (9.3.5)] that

$$3 \leqslant \lambda^{\mathfrak{a}}_{\mathfrak{m}}(M) := \inf\bigl\{\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) \bigm| \mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Var}(\mathfrak{a})\bigr\}.$$

As R is catenary, each $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Var}(\mathfrak{a})$ satisfies $\operatorname{ht}((\mathfrak{a}+\mathfrak{p})/\mathfrak{p}) \geqslant \dim(R/\mathfrak{p}) - \dim(R/\mathfrak{a}) \geqslant \dim(R/\mathfrak{p}) - 1$ and it follows that for each such $\mathfrak{p}$ it holds $\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \operatorname{ht}((\mathfrak{a} + \mathfrak{p})/\mathfrak{p}) \geqslant \operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) - 1 \geqslant \lambda^{\mathfrak{a}}_{\mathfrak{m}}(M) - 1 \geqslant 3 - 1 = 2$.

Therefore $\lambda_{\mathfrak{a}}(M) := \min\{\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \operatorname{ht}((\mathfrak{a} + \mathfrak{p})/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Var}(\mathfrak{a})\} \geqslant 2$. As R is a homomorphic image of a local Gorenstein ring it is universally catenary and all its formal fibers are CM. Hence, by [BS, (9.6.7)] we get

$$f_{\mathfrak{a}}(M) := \inf\bigl\{i \in \mathbb{N}_0 \bigm| H^i_{\mathfrak{a}}(M) \text{ is not finitely generated}\bigr\} = \lambda_{\mathfrak{a}}(M) \geqslant 2.$$

In particular, $H^1_{\mathfrak{a}}(M)$ is finitely generated. So the four-term exact sequence (cf. [BS, (2.2.4)(i)])

$$0 \to H^0_{\mathfrak{a}}(M) \to M \to D_{\mathfrak{a}}(M) \to H^1_{\mathfrak{a}}(M) \to 0$$

shows that $N := D_{\mathfrak{a}}(M)$ is finitely generated.

Moreover, $H^i_{\mathfrak{a}}(N) = 0$ for $i = 0, 1$ (cf. [BS, (2.2.8)(iv)]). Therefore $\operatorname{grade}_N(\mathfrak{a}) \geqslant 2$. As $\mathfrak{a} \subseteq \mathfrak{m}$, it follows that $H^i_{\mathfrak{m}}(N) = 0$ for $i = 0, 1$.

Our next aim is to show that $H^2_{\mathfrak{m}}(N) = H^2_{\mathfrak{m}}(D_{\mathfrak{a}}(M))$ is of finite length. To this end let $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. If $\mathfrak{p} \in \operatorname{Var}(\mathfrak{a})$ then $\dim(R/\mathfrak{p}) = 1$ and $\operatorname{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \geqslant \operatorname{grade}_N(\mathfrak{a}) \geqslant 2$ and hence $\operatorname{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) \geqslant 3$.

If $\mathfrak{p} \notin \operatorname{Var}(\mathfrak{a})$, then $H^0_{\mathfrak{a}}(M)_{\mathfrak{p}} = H^1_{\mathfrak{a}}(M)_{\mathfrak{p}} = 0$ and hence the above four-term exact sequence yields $N_{\mathfrak{p}} \cong M_{\mathfrak{p}}$, so that $\operatorname{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) = \operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) \geqslant \lambda^{\mathfrak{a}}_{\mathfrak{m}}(M) \geqslant 3$.

Altogether we obtain $\lambda_{\mathfrak{m}}(N) := \inf\{\operatorname{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}\} \geqslant 3$. So, by [BS, (9.6.7)] we get $f_{\mathfrak{m}}(N) := \inf\{i \in \mathbb{N} \mid H^i_{\mathfrak{m}}(N) \text{ is not finitely generated}\} = \lambda_{\mathfrak{m}}(M) \geqslant 3$. Therefore $H^2_{\mathfrak{m}}(N)$ is finitely generated and thus of finite length.

By our hypothesis the finitely generated R-modules $H^0_{\mathfrak{a}}(M)$ and $H^1_{\mathfrak{a}}(M)$ are both of dimension $\leqslant 1$. Another use of the above four-term exact sequence now yields that $H^i_{\mathfrak{m}}(N) \cong H^i_{\mathfrak{m}}(M)$ for all $i \geqslant 3$. $\square$

Lemma 3.5. *Let the notations and the assumptions be as in Lemma 3.4. Then $H^2_{\mathfrak{m}}(D_{\mathfrak{a}}(M)) = 0$ if and only if $\mathfrak{m} \notin \mathrm{Att}_R(H^2_{\mathfrak{m}}(M))$.*

Proof. Set $N := D_{\mathfrak{a}}(M)$ and let $\overline{M} := M/H^0_{\mathfrak{a}}(M)$. As $\dim(H^0_{\mathfrak{a}}(M)) \leqslant \dim(R/\mathfrak{a}) \leqslant 1$, we have $H^2_{\mathfrak{m}}(M) \cong H^2_{\mathfrak{m}}(\overline{M})$.

Assume first that $\mathfrak{m} \notin \mathrm{Att}_R(H^2_{\mathfrak{m}}(M))$. Then there is some $x \in \mathfrak{m}$ which avoids all members of $\mathrm{Att}_R(H^2_{\mathfrak{m}}(M))$ and hence multiplication by x on $H^2_{\mathfrak{m}}(\overline{M})$ is surjective. Since $\dim(H^1_{\mathfrak{a}}(M)) \leqslant \dim(R/\mathfrak{a}) \leqslant 1$, it holds $H^2_{\mathfrak{m}}(H^1_{\mathfrak{a}}(M)) = 0$. Therefore, the short exact sequence (cf. [BS, (2.2.4)(i)])

$$0 \to \overline{M} \to N \to H^1_{\mathfrak{a}}(M) \to 0$$

gives rise to an epimorphism $H^2_{\mathfrak{m}}(\overline{M}) \twoheadrightarrow H^2_{\mathfrak{m}}(N)$. Hence, multiplication by x on $H^2_{\mathfrak{m}}(N)$ is surjective. As $H^2_{\mathfrak{m}}(N)$ is finitely generated, Nakayama yields $H^2_{\mathfrak{m}}(N) = 0$.

Conversely, assume that $H^2_{\mathfrak{m}}(N) = 0$. Then, the above exact sequence gives rise to an epimorphism $H^1_{\mathfrak{m}}(H^1_{\mathfrak{a}}(M)) \twoheadrightarrow H^2_{\mathfrak{m}}(\overline{M})$. Let $x \in \mathfrak{m}$ be such that $\mathrm{rad}(\mathfrak{a} + xR) = \mathfrak{m}$. Then $H^1_{\mathfrak{m}}(H^1_{\mathfrak{a}}(M)) \cong H^1_{xR}(H^1_{\mathfrak{a}}(M))$, so that multiplication by x on $H^1_{\mathfrak{m}}(H^1_{\mathfrak{a}}(M))$ is an epimorphism. Hence multiplication by x on $H^2_{\mathfrak{m}}(\overline{M}) \cong H^2_{\mathfrak{m}}(M)$ is an epimorphism, so that $\mathfrak{m} \notin \mathrm{Att}_R(H^2_{\mathfrak{m}}(M))$. □

For each $j \in \{0, \dots, d\}$ let $M^{[j]} := M/M_{[j]}$, where $M_{[j]} \subset M$ is the largest submodule of M whose dimension is $\leqslant j$. Keep in mind that we can write $M_{[j]}$ as the torsion submodule $\Gamma_{\mathfrak{B}}(M) := \{m \in M \mid \exists \mathfrak{b} \in \mathfrak{B}$ such that $\mathfrak{b}m = 0\}$ of M with respect to the multiplicatively closed set of ideals $\mathfrak{B} := \{\mathfrak{b} = \text{ideal in } R \mid \dim(R/\mathfrak{b}) \leqslant j\}$.

Lemma 3.6. *Suppose that $d \geqslant 3$ and let $p(M) = 1$. Let $\mathfrak{a} := (0 :_R H^2_{\mathfrak{m}}(M))$. Then the following statements are equivalent:*

(i) *$H^i_{\mathfrak{m}}(M) = 0$ for all $i = 3, \dots, d-1$ and $K^2(M)$ is* 0 *or* CM *of dimension* 1.
(ii) *$D_{\mathfrak{a}}(M^{[1]})$ is a (finitely generated)* CM *module.*

Proof. As $\dim(M_{[1]}) \leqslant 1$, it is clear that $H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(M^{[1]})$ for all $i \geqslant 2$ and that $K^2(M) \cong K^2(M^{[1]})$. Observe that by our hypotheses and by Remark 3.2 we have $\dim(R/\mathfrak{a}) \leqslant 1$ and hence also $\dim(K^2(M)) \leqslant 1$.

Our next claim is that $\ell(H^1_{\mathfrak{m}}(M^{[1]})) < \infty$. Suppose that this is not the case. Then $\dim H^1_{\mathfrak{m}}(M^{[1]}) = 1$. Hence $\dim(R/\mathfrak{p}) = 1$ for some $\mathfrak{p} \in \mathrm{Att}_R H^1_{\mathfrak{m}}(M^{[1]})$. So, $\mathfrak{p} \in \mathrm{Ass}_R(M^{[1]})$ by [BS, 11.3.3], this is a contradiction as $\mathrm{Ass}_R M^{[1]} = \{\mathfrak{q} \in \mathrm{Ass}_R M \mid \dim(R/\mathfrak{q}) > 1\}$.

So, by Lemma 3.4 the module $N := D_{\mathfrak{a}}(M^{[1]})$ is finitely generated and $H^i_{\mathfrak{m}}(N) \cong H^i_{\mathfrak{m}}(M^{[1]}) \cong H^i_{\mathfrak{m}}(M)$ for all $i \geqslant 3$.

Therefore N is CM if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all $i \in \{3, \dots, d-1\}$ and $K^2(N) = 0$. It thus remains to show that $K^2(N) = 0$ if and only if $K^2(M)$ is of depth > 0 or equivalently, if and only if $\mathfrak{m} \notin \mathrm{Att}_R(H^2_{\mathfrak{m}}(M))$. But this is clear by Lemma 3.5. □

Proposition 3.7. *Suppose that $d \geqslant 3$ and let $p(M) = 1$. Let $\mathfrak{a} := (0 :_R H^2_{\mathfrak{m}}(M))$. Then the following statements are equivalent:*

(i) *M is* CCM.
(ii) *$H^i_{\mathfrak{m}}(M) = 0$ for all $i = 3, \dots, d-1$ and $K^2(M)$ is either* 0 *or* CM *of dimension* 1.
(iii) *$D_{\mathfrak{a}}(M^{[1]})$ is a (finitely generated)* CM *module.*

Proof. (i) ⇒ (ii): Let $x \in \mathfrak{m}$ be a strict f-element with respect to M. Since M is CCM so is $M_1 := M/xM$ by Theorem 2.5. As $p(M) = 1$, we get by Lemma 3.3 that M_1 is generalized CM. By Corollary 2.7, we have $K^i(M_1) = 0$ for all $i = 2, \dots, d-2$. By Lemma 2.4(b) we thus see that $K^i(M) = 0$ for all $i = 3, \dots, d-1$ and that $x \in \mathfrak{m}$ is $K^2(M)$-regular. Since $p(M) = 1$, we have $\dim K^2(M) \leqslant 1$ by Remark 3.2. Therefore $K^2(M)$ is 0 or CM of dimension 1.

(ii) $\Rightarrow$ (i): Let $x \in \mathfrak{m}$ and M_1 be as above. Assume first that $d = 3$. Since $\operatorname{depth}(K^2(M)) > 0$ we have $(0 :_{K^2(M)} x) = 0$. If we apply the sequence of Lemma 2.4(a) with $i = 2$ we obtain $K(M)/xK(M) \cong K(M_1)$. As $\dim M_1 = 2$, the module $K(M_1)$ is CM. As x is $K(M)$-regular it follows that $K(M)$ is CM.

Now, let $d > 3$. Since $K^i(M) = 0$ for all $i = 3, \dots, d-1$, it follows by the exact sequences of Lemma 2.4(a) that $K^i(M_1) = 0$ for all $i = 3, \dots, d-2$. As $K^2(M)$ is 0 or CM of dimension 1, we have $(0 :_{K^2(M)} x) = 0$. Since $d > 3$, we have $K^3(M) = 0$. Therefore another use of the mentioned exact sequences for $i = 2$ gives that $K^2(M_1) = 0$. Note that M_1 is generalized CM of dimension $d-1$ by Lemma 3.3. Hence M_1 is CCM by Corollary 2.7, and so M is CCM by Theorem 2.5.

(ii) $\Leftrightarrow$ (iii): This is clear by Lemma 3.6. $\square$

The main result of this section is the extension of the equivalence (i) $\Leftrightarrow$ (ii) of Proposition 3.7 which applies for all possible values of $p(M)$ and which again involves strict f-sequences. To pave the way to this, we first prove two auxiliary results.

Lemma 3.8. *Let $p(M) =: k > 0$ and let $x_1, \dots, x_{k-1}$ be a strict* f*-sequence with respect to M. Set $M_i = M/\sum_{j=1}^{i} x_j M$ for all $i = 0, \dots, k-1$. Then, the following statements are equivalent*:

(i) *M is* CCM *with $k \leqslant d-2$.*
(ii) *$K^i(M) = 0$ for all $i = k+2, \dots, d-1$ and $K^2(M_{k-1})$ is* 0 *or* CM *of dimension* 1.

Proof. (i) $\Rightarrow$ (ii): We proceed by induction on k. The case $k = 1$ follows by Proposition 3.7. So, let $k \geqslant 2$ and note that M_1 is CCM by Theorem 2.5. By Lemma 3.3 we have $p(M_1) = k-1 \leqslant \dim M_1 - 2$. As $x_2, \dots, x_{k-1}$ is a strict f-sequence with respect to M_1, we get by induction that $K^2(M_{k-1}) = K^2(M_1/\sum_{j=2}^{k-1} x_j M_1)$ is 0 or CM of dimension 1 and that $K^i(M_1) = 0$ for all $i = k+1, \dots, d-2$. By Lemma 2.4(b) the latter implies that $K^i(M) = 0$ for all $i = k+2, \dots, d-1$.

(ii) $\Rightarrow$ (i): We proceed again by induction on k. The case $k = 1$ is immediately clear by Proposition 3.7. So, let $k \geqslant 2$. Observe that $(x_2, \dots, x_{k-1})$ is a strict f-sequence with respect to M_1. By Lemma 3.3 we also have $p(M_1) = k-1$. As $K^i(M) = 0$ for all $i = k+2, \dots, d-1$, the exact sequences of Lemma 2.4(a), applied with $x := x_1$, imply that $K^i(M_1) = 0$ for all $i = k+2, \dots, d-2$.

Our next aim is to show that $K^{k+1}(M_1) = 0$. Keep in mind that x_i is a strict f-element with respect to M_{i-1} and that $M_{i-1}/x_i M_{i-1} \cong M_i$, for each $i = 1, \dots, k-1$. Observe also that $K^2(M_{k-1})$ is 0 or CM of dimension 1 by assumption (ii).

Assume first that $K^2(M_{k-1}) = 0$. Then, applying Lemma 2.4(b) with $j+2$ instead of i, with M_{k-j} instead of M and with x_{k-j} instead of x for $j = 1, \dots k-1$ we inductively get that $K^{2+j}(M_{k-j-1}) = 0$ so that finally $K^{k+1}(M) = 0$. By the hypothesis (ii) we also have $K^{k+2}(M) = 0$. So, the sequence of Lemma 2.4(a), applied with $i = k+1$ and $x = x_1$ gives indeed $K^{k+1}(M_1) = 0$ in this case, as requested.

Now, we consider the case where $K^2(M_{k-1})$ is CM of dimension 1. In this situation we have $\operatorname{depth}(K^2(M_{k-1})) > 0$. Applying 2.4(d) for $i = 2, \dots, k$ to M_{k-i} instead of M and x_{k-i+1} instead of x, we inductively get that $\operatorname{depth}(K^{i+1}(M_{k-i})) > 0$ and hence in particular that $\operatorname{depth}(K^{k+1}(M)) > 0$. Since x_1 is an f-element with respect to $K^{k+1}(M)$, this implies that $0 :_{K^{k+1}(M)} x_1 = 0$. By (ii) we have $K^{k+2}(M) = 0$ and hence on use of the exact sequence of Lemma 2.4(a) with $i = k+1$ and $x = x_1$ it follows that again $K^{k+1}(M_1) = 0$ in this second case.

Moreover, by (ii) the module $K^2(M_1/\sum_{j=2}^{k-1} x_j M_1) = K^2(M_{k-1})$ is 0 or CM of dimension 1. So, M_1 satisfies the hypothesis of induction. Therefore M_1 is CCM and $k-1 = p(M) - 1 = p(M_1) \leqslant d-3$, so that $p(M) = k \leqslant d-2$. Since $k \geqslant 2$, we have $d \geqslant 4$ and thus M is CCM by Theorem 2.5. $\square$

Lemma 3.9. *Let $d \geqslant 3$ and let $x_1, \dots, x_{d-3}$ be a strict* f*-sequence with respect to M. Then, the following statements are equivalent*:

(i) *M is* CCM *with $p(M) = d-1$.*
(ii) *$K^2(M/\sum_{j=1}^{d-3} x_j M)$ is of dimension* 2 *and of positive depth.*

Proof. Set $M_i := M/\sum_{j=1}^{i} x_j M$ for all $i = 1, \dots, d-3$.

(i) $\Rightarrow$ (ii): By Corollary 2.6 we have $\operatorname{depth}(K^2(M_{d-3})) > 0$. Note that $\dim K^i(M) \leqslant i$ for all i (see [Sc3, Proposition 2.3(a)]). Since $p(M) = d-1$, Remark 3.2 implies that $\dim(K^{d-1}(M)) = d-1$. Assume first that $d = 3$. Then $\dim(K^2(M)) = 2$ and the result follows.

So, let $d > 3$. Then $\dim(K^{d-1}(M)) = d-1$. On use of Lemma 2.4(e) with $i = d-j$, M_{j-1} instead of M, x_j instead of x for $j = 2, \dots, d-2$, we inductively get that $\dim K^{d-j}(M_{j-1}) = d-j$. Hence in particular $\dim K^2(M_{d-3}) = 2$.

(ii) $\Rightarrow$ (i): By Corollary 2.6, the module M is CCM. By (ii) we have $\dim K^2(M_{d-3}) = 2$. By Lemma 2.4(e) applied with M_{d-i-1} instead of M and x_{d-i} instead of x for $i = 1, \dots, d-2$ we inductively get that $\dim K^{i+1}(M_{d-i-2}) = i+1$, so that $\dim(K^{d-1}(M)) = d-1$. Therefore by Remark 3.2 we have $p(M) = d-1$. □

Now, we are ready to give the announced main result of the present section.

Theorem 3.10. *Assume that* $\dim M = d \geqslant 3$, $p(M) = k > 0$ *and that* $x_1, \dots, x_{k-1}$ *is a strict* f*-sequence with respect to* M. *For all* $i = 1, \dots, k-1$ *set* $M_i := M/\sum_{j=1}^{i} x_j M$.

(a) *If* $k \leqslant d-2$, *then* M *is* CCM *if and only if* $K^i(M) = 0$ *for all* $i = k+2, \dots, d-1$ *and* $K^2(M_{k-1})$ *is* 0 *or* CM *of dimension* 1.
(b) *If* $k = d-1$ *then* M *is* CCM *if and only if* $K^2(M_{d-3})$ *is of dimension* 2 *and of positive depth.*

Proof. The proof is immediate by Lemma 3.8 and Lemma 3.9. □

4. Non-canonical Cohen–Macaulay loci

Definition 4.1. The *non-canonical Cohen–Macaulay locus* (non-CCM locus for short) of M, denoted by $\operatorname{nCCM}(M)$, is defined by

$$\operatorname{nCCM}(M) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_{\mathfrak{p}} \text{ is not CCM}\}.$$

Before describing the non-CCM locus of M, we recall the notion of pseudo support introduced in [BS1].

Definition 4.2. The *i-th pseudo support* of M, denoted by $\operatorname{Psupp}_R^i(M)$, is defined by

$$\operatorname{Psupp}_R^i(M) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid H_{\mathfrak{p}R_{\mathfrak{p}}}^{i-\dim(R/\mathfrak{p})}(M_{\mathfrak{p}}) \neq 0\}.$$

Remark 4.3. (a) Since R is a quotient of a Gorenstein ring

$$\operatorname{Psupp}_R^i(M) = \operatorname{Var}(\operatorname{Ann}_R H_{\mathfrak{m}}^i(M)) = \operatorname{Supp}_R K^i(M),$$

which is a closed subset of $\operatorname{Spec}(R)$ (see [BS1]).

(b) Keep in mind that

$$\operatorname{Ass}_R K(M) = \operatorname{Att}_R H_{\mathfrak{m}}^d(M) = \{\mathfrak{q} \in \operatorname{Ass}_R M \mid \dim(R/\mathfrak{q}) = d\},$$

so that $K(M)$ is equidimensional. Therefore, by [CNN, Corollary 3.2] we have

$$\operatorname{nCM}(K(M)) = \bigcup_{i=0,\dots,d-1} \operatorname{Psupp}^i(K(M)).$$

(c) Observe also that

$$\operatorname{Supp}_R K(M) = \{\mathfrak{p} \in \operatorname{Supp}_R M \mid \dim(R/\mathfrak{p}) + \dim M_{\mathfrak{p}} = d\}.$$

In particular $K(M)_{\mathfrak{p}} \cong K(M_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Supp}_R K(M)$.

Proposition 4.4. *If* $\dim(R/\mathfrak{q}) = d$ *or* $\dim(R/\mathfrak{q}) \leqslant 3$ *for all* $\mathfrak{q} \in \min \operatorname{Ass}_R M$ *then*

$$\operatorname{nCCM}(M) = \operatorname{nCM}(K(M)) = \bigcup_{i=0,\dots,d-1} \operatorname{Psupp}_R^i(K(M)),$$

which is a closed subset of $\operatorname{Spec}(R)$*. In particular, if* $\dim M \leqslant 4$ *then* $\operatorname{nCCM}(M)$ *is closed.*

Proof. By Remark 4.3(b) it suffices to show that $\operatorname{nCCM}(M) = \operatorname{nCM}(K(M))$. First, let $\mathfrak{p} \in \operatorname{Supp}_R K(M)$. It follows by Remark 4.3(c) that $K(M_{\mathfrak{p}}) \cong (K(M))_{\mathfrak{p}}$. Therefore $\mathfrak{p} \in \operatorname{nCCM}(M)$ if and only if $(K(M))_{\mathfrak{p}}$ is not CM, hence if and only if $\mathfrak{p} \in \operatorname{nCM}(K(M))$. This shows that $\operatorname{Supp}_R K(M) \cap \operatorname{nCCM}(M) = \operatorname{nCM}(K(M))$.

Next, assume that $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Supp}_R K(M)$. Then $\dim(R/\mathfrak{q}) < d$ for all $\mathfrak{q} \in \operatorname{Ass} M$ with $\mathfrak{q} \subseteq \mathfrak{p}$. In particular $\mathfrak{p} \neq \mathfrak{m}$. So by our hypothesis we have $\dim M_{\mathfrak{p}} \leqslant 2$. Hence $K(M_{\mathfrak{p}})$ is CM, so that $\mathfrak{p} \notin \operatorname{nCCM}(M)$. $\square$

Reminder 4.5. The notion of dimension filtration was introduced by P. Schenzel [Sc2]. A sequence of submodules $H^0_{\mathfrak{m}}(M) = M_0 \subset M_1 \subset \cdots \subset M_t = M$ is called a *dimension filtration* of M if M_i is the largest submodule $(M_{i+1})_{[\dim(M_{i+1})-1]}$ of M_{i+1} such that $\dim M_i < \dim M_{i+1}$ for all $i = 0, \dots, t-1$. Note that there exists precisely one dimension filtration of M.

Proposition 4.6. *Let* $H^0_{\mathfrak{m}}(M) = M_0 \subset M_1 \subset \cdots \subset M_t = M$ *be the dimension filtration of* M*. Set* $d_k = \dim M_k$*. Then*

$$\operatorname{nCCM}(M) \subseteq \bigcup_{\substack{k=1,\dots,t\\ i=0,\dots,d_k-1}} \operatorname{Psupp}^i(K(M_k)) = \bigcup_{k=1,\dots,t} \operatorname{nCM}(K(M_k)).$$

Proof. We proceed by induction on t. If $t = 0$ then $\operatorname{nCCM}(M) = \emptyset$ and the result is true. Let $t = 1$. Then $\dim(R/\mathfrak{p}) = d$ for all $\mathfrak{p} \in \operatorname{Ass}_R M \setminus \{\mathfrak{m}\}$. Therefore Proposition 4.4 yields $\operatorname{nCCM}(M) = \operatorname{nCM}(K(M)) = \bigcup_{i=0,\dots,d} \operatorname{Psupp}^i(K(M))$. Thus, the result is true for $t = 1$.

Now, let $t > 1$ and let $\mathfrak{p} \in \operatorname{nCCM}(M)$. Assume first that $\mathfrak{p} \in \operatorname{Supp}_R K(M)$. By Remark 4.3(c) we have $K(M_{\mathfrak{p}}) \cong (K(M))_{\mathfrak{p}}$, and hence $\mathfrak{p} \in \operatorname{nCM}(K(M)) \subseteq \bigcup_{i=0,\dots,d} \operatorname{Psupp}^i(K(M))$, by another use of Remark 4.3(c).

Suppose now, that $\mathfrak{p} \notin \operatorname{Supp}_R K(M)$. Then $\mathfrak{q} \nsubseteq \mathfrak{p}$ for all $\mathfrak{q} \in \operatorname{Ass}_R M$ with $\dim(R/\mathfrak{q}) = d$ by Remark 4.3(c). Since

$$\operatorname{Ass}_R(M/M_{t-1}) = \{\mathfrak{q} \in \operatorname{Ass}_R M \mid \dim(R/\mathfrak{q}) = d\},$$

we have $\mathfrak{p} \notin \operatorname{Supp}(M/M_{t-1})$. So, from the exact sequence

$$0 \to (M_{t-1})_{\mathfrak{p}} \to M_{\mathfrak{p}} \to (M/M_{t-1})_{\mathfrak{p}} \to 0$$

we get $M_{\mathfrak{p}} \cong (M_{t-1})_{\mathfrak{p}}$. Hence $\mathfrak{p} \in \operatorname{nCCM}(M_{t-1})$. So by induction

$$\mathfrak{p} \in \bigcup_{k=1,\dots,t-1} \operatorname{nCM}(K(M_k)) = \bigcup_{\substack{k=1,\dots,t-1\\ i=0,\dots,d_k-1}} \operatorname{Psupp}^i(K(M_k)). \qquad \square$$

Finally, we give an example to show that for $d \geqslant 5$ the set $\mathrm{nCCM}(M)$ need not be closed, indeed even not stable under specialization. We first prove the following lemma.

Lemma 4.7. *Let K be a field, let n be an integer such that $n \geqslant 3$. Consider the polynomial ring $U = K[x_1, \dots, x_{n+2}]$, furnished with its standard grading. Then there is a graded prime ideal $\mathfrak{t} \subseteq U$ of height 2 such that $\mathfrak{t} \cap U_1 = 0$ and such that $A := U/\mathfrak{t}$ is a normal domain (of dimension n) with homogeneous maximal ideal $A_+ = U_+/\mathfrak{t}$ and*

$$H^i_{A_+}(A) = \begin{cases} 0, & \text{if } i \neq n-1, n \\ A/A_+, & \text{if } i = n-1. \end{cases}$$

Proof. See [EG, Theorem 4.13] or [MNP, Theorem 2.3 and Remark 2.4(ii)]. □

Notation and Construction 4.8. Let $n, U, \mathfrak{t}$ and A be as in Lemma 4.7. Consider the polynomial ring $S := U[x_{n+3}] = K[x_1, \dots, x_{n+3}]$ and set $R := (S/\mathfrak{t}S \cap x_{n+3}S)_{S_+}$, $\mathfrak{m} := S_+R$, $\mathfrak{r} := x_{n+3}R$, $\mathfrak{q} := \mathfrak{t}R$ and $\mathfrak{p} := U_+R$.

Proposition 4.9. *$(R, \mathfrak{m})$ is a reduced local ring of dimension $n+2$, essentially of finite type over K with $R/\mathfrak{m} \cong K$ and $\dim_K(\mathfrak{m}/\mathfrak{m}^2) = n+3$. Moreover,*

(a) $\min \mathrm{Ass}\, R = \{\mathfrak{r}, \mathfrak{q}\}$ *and* $\mathfrak{p} \in \mathrm{Var}(\mathfrak{q}) \setminus \mathrm{Var}(\mathfrak{r})$.
(b) *$R/\mathfrak{r}$ is regular local of dimension $n+2$, $R/\mathfrak{q}$ is normal of dimension $n+1$ and $R_\mathfrak{p}$ is normal of dimension n.*
(c) $H^i_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}) = \begin{cases} 0, & \text{if } i \neq n-1, n \\ K(\mathfrak{p}), & \text{if } i = n-1. \end{cases}$
(d) $\mathrm{nCCM}(R) = \{\mathfrak{p}\}$.

Proof. The claims in the preamble of our proposition follow immediately by the previous construction.

(a) This is also an easy consequence of our construction.
(b) Observe that according to our construction

$$R/\mathfrak{r} \cong U_{U_+} \cong K[x_1, \dots, x_{n+2}]_{(x_1,\dots,x_{n+2})} \quad \text{and} \quad R/\mathfrak{q} \cong A[x_{n+3}]_{(A_+, x_{n+3})}.$$

Since $\mathfrak{p} \in \mathrm{Var}(\mathfrak{q}) \setminus \mathrm{Var}(\mathfrak{r})$, we have

$$R_\mathfrak{p} \cong (R/\mathfrak{q})_\mathfrak{p} \cong A[x_{n+3}]_{A_+.A[x_{n+3}]}.$$

From this, all claims in statement (b) are immediate.
(c) The last isomorphism in the previous paragraph and the A-flatness of $A[x_{n+3}]_{A_+.A[x_{n+3}]}$ yield

$$\begin{aligned} H^i_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}) &\cong H^i_{A_+.A[x_{n+3}]_{A_+.A[x_{n+3}]}}\big(A[x_{n+3}]_{A_+.A[x_{n+3}]}\big) \\ &\cong H^i_{A_+}(A) \otimes_A A[x_{n+3}]_{A_+.A[x_{n+3}]} \\ &\cong H^i_{A_+}(A) \otimes_A R_\mathfrak{p}. \end{aligned}$$

As $A_+R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$, this proves our claim.
(d) According to statement (c), the ring $R_\mathfrak{p}$ is generalized CM but not CCM. Therefore $\mathfrak{p} \in \mathrm{nCCM}(R)$. By statement (b), the unmixed part $U(R) := R^{[\dim R - 1]}$ of R is given by $U(R) = R/\mathfrak{r}$ and as $R/\mathfrak{r}$ is regular, it follows that $K(R) = K(U(R)) = R/\mathfrak{r}$ and hence $\mathrm{Var}(\mathfrak{r}) \cap \mathrm{nCCM}(R) = \emptyset$.

It remains to show that $K(R_{\mathfrak{s}})$ is CM for all $\mathfrak{s} \in \operatorname{Spec}(R) \setminus (\operatorname{Var}(\mathfrak{r}) \cup \{\mathfrak{p}\})$. Fix such an $\mathfrak{s}$ and consider the canonical map $A \to R$. As $\operatorname{Var}(A_+R) = \{\mathfrak{p}, \mathfrak{m}\}$, we must have that $\mathfrak{n} := \mathfrak{s} \cap A \subset A_+$ and $\mathfrak{n} \neq A_+$, so that $A_{\mathfrak{n}}$ is CM. In view of the last isomorphism in the proof of (b), the ring $R_{\mathfrak{s}}$ is isomorphic to a localization of $A_{\mathfrak{n}}[x_{n+3}]$ and hence is CM, so that $K(R_{\mathfrak{s}})$ is CM. □

Corollary 4.10. *For each field K and each integer $d \geqslant 5$ there is a reduced local ring $(R, \mathfrak{m})$ of dimension d which is essentially of finite type over K, with $R/\mathfrak{m} \cong K$ and such that* $\operatorname{nCCM}(R)$ *is not stable under specialization.*

Proof. Apply Proposition 4.9 with $n = d - 2$. □

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